

THE SPACE OF COMPACT SELF-SHRINKING SOLUTIONS TO THE LAGRANGIAN MEAN CURVATURE FLOW IN \mathbb{C}^2

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ABSTRACT. Let $F_n : (\Sigma, h_n) \rightarrow \mathbb{C}^2$ be a sequence of conformally immersed Lagrangian self-shrinkers with a uniform area upper bound to the mean curvature flow, and suppose that the sequence of metrics $\{h_n\}$ converges smoothly to a Riemannian metric h . We show that a subsequence of $\{F_n\}$ converges smoothly to a branched conformally immersed Lagrangian self-shrinker $F_\infty : (\Sigma, h) \rightarrow \mathbb{C}^2$. When the area bound is less than 16π , the limit F_∞ is an embedded torus. When the genus of Σ is one, we can drop the assumption on convergence $h_n \rightarrow h$. When the genus of Σ is zero, we show that there is no branched immersion of Σ as a Lagrangian shrinker, generalizing the rigidity result of [21] in dimension two by allowing branch points.

1. INTRODUCTION

Let Σ be an orientable smooth m dimensional manifold without boundary and let F be an immersion of Σ into \mathbb{R}^{m+k} . The smooth mapping F is called a *self-shrinker* if it satisfies the equation

$$(1.1) \quad \vec{H} = -F^\perp,$$

where \vec{H} is the mean curvature vector and F^\perp is the normal component of the position vector F . A self-shrinker gives rise to a solution, via the scaling $F_t = \sqrt{1-2t} F$ for $0 \leq t < 1/2$, of the mean curvature flow (MCF) in \mathbb{R}^{m+k} :

$$\left(\frac{\partial F_t}{\partial t} \right)^\perp = \vec{H}(F_t).$$

MCF typically develops singularities in finite time. By parabolic rescaling at a singularity, Huisken's monotonicity formula [15] shows that a subsequence of the mean curvature flow converges to a self-shrinker when the singularity is of type I. Ultimately, the classification of self-shrinkers would be important in understanding the MCF.

When $m+k = 2m$, we can identify \mathbb{R}^{2m} with \mathbb{C}^m and consider the Lagrangian immersions. It is shown in [22] that the Lagrangian condition is preserved along the MCF. Therefore, the shrinkers arise from Lagrangian MCF are Lagrangian submanifolds. When $m = 1$, the Lagrangian condition is automatically satisfied by smooth curves and self shrinking solutions are studied in [3].

In this paper, we will restrict our attention to compact Lagrangian self shrinkers in \mathbb{C}^2 . For arbitrary m , Smoczyk [21] showed that there is no Lagrangian self-shrinking

Date: June 24, 2014.

The first author is partially supported by an NSERC grant.

immersion in \mathbb{C}^m with first Betti number equals zero. In particular, there is no immersed Lagrangian self shrinking sphere in \mathbb{C}^2 . To establish compactness properties of the moduli space of compact Lagrangian shrinkers, it is crucial to generalize Smoczyk's result to *branched* Lagrangian immersions as the limit of a sequence of immersions may not be an immersion anymore.

Indeed, the rigidity holds for branched Lagrangian shrinking spheres in \mathbb{C}^2 , we have:

Theorem 1.1. *There does not exist any branched conformal Lagrangian self-shrinking sphere in \mathbb{C}^2 .*

At any immersed point, the mean curvature form $\alpha_H = \iota_{\bar{H}}\omega$ of the Lagrangian shrinker satisfies a pair of differential equations (2.4) and (2.5) that form a first order elliptic system. The key ingredient in the proof of Theorem 1.1 is to show that α_H extends smoothly across the branch points. The self-shrinker equation (1.1) yields a L^∞ bound on α_H and this is useful to show that α_H satisfies the first order elliptic system as a weak solution distributionally. Smoothness of the extended mean curvature form then follows from the elliptic theory.

The main application of Theorem 1.1 is to derive compactness results of compact Lagrangian self-shrinkers.

Let Σ be a fixed compact oriented smooth surface and $F_n : \Sigma \rightarrow \mathbb{C}^2$ be a sequence of Lagrangian immersed self-shrinkers in \mathbb{C}^2 . Let h_n be the Riemannian metric on Σ which is conformal to the pull back metric $F_n^*\langle \cdot, \cdot \rangle$ on Σ such that either it has constant Gauss curvature -1 if the genus of Σ is greater than one, or (Σ, h_n) is $\mathbb{C}/\{1, a + bi\}$ with the flat metric, where $-\frac{1}{2} < a \leq \frac{1}{2}$, $b \geq 0$, $a^2 + b^2 \geq 1$ and $a \geq 0$ whenever $a^2 + b^2 = 1$. It is well known that the Moduli space of the conformal structures on Σ are parametrized by metrics of the above form.

We now state our compactness result.

Theorem 1.2. *Let $F_n : (\Sigma, h_n) \rightarrow \mathbb{C}^2$ be a sequence of conformally immersed Lagrangian self-shrinkers with a uniform area upper bound Λ . Suppose that the sequence of metrics $\{h_n\}$ converges smoothly to a Riemannian metric h on Σ . Then there is a subsequence of $\{F_n\}$ that converges smoothly to a branched conformally immersed Lagrangian self-shrinkers $F_\infty : (\Sigma, h) \rightarrow \mathbb{C}^2$.*

Note that there is a universal positive lower bound on the extrinsic diameters for two dimensional compact shrinkers (cf. section 4). The limit F_∞ cannot be a constant map.

The proof of Theorem 1.2 uses the observation that self-shrinkers are in fact minimal immersions into (\mathbb{C}^2, G) , where G is a metric on \mathbb{C}^2 conformal to the Euclidean metric. The advantage of this viewpoint is that we are then able to use the bubble tree convergence of harmonic maps developed in [19]. In particular, Theorem 1.1 shows that no bubble is formed during the process and thus the convergence is smooth.

It is interesting to compare Theorem 1.2 with the compactness results of Colding-Minicozzi [7] on self-shrinkers in \mathbb{R}^3 and of Choi-Schoen [9], Fraser-Li [12] on embedded minimal surfaces in three dimensional manifold N (with or without boundary) with nonnegative Ricci curvature. In their cases, they use the local singular compactness theorem (see for example Proposition 2.1 of [7]), which says that for any sequence of

embedded minimal surfaces Σ_n in N with a uniform (global or local) area or genus upper bound, there is an embedded minimal surface Σ such that a subsequence of Σ_n converges smoothly and locally (with finite multiplicities) to Σ away from finitely many points in Σ . A removable singularity theorem, which is based on the maximum principle and is true only in the codimension one case, is needed to prove the local singular compactness theorem. Hence similar statement is not available in higher codimensions. For embedded minimal surfaces in a 3-manifold, Colding and Minicozzi have proven deep compactness results ([8] and the reference therein).

In the case of arbitrary codimension, the regularity of the limit of a sequence of minimal surfaces (as a stationary varifold given by Geometric Measure Theory) is a subtle issue. Not much is known except for some special cases, such as Gromov's compactness theorem on pseudo holomorphic curves [13].

In our situation, we do not assume embeddedness. In fact, if we assume that F_n is a Lagrangian embedding, then by a result of Whitney ([25], see also [2]), Σ must be a torus as the self-intersection number $\Sigma \cdot \Sigma$ is twice of the genus of Σ minus 2. When we treat the immersions F_n as conformal harmonic mappings, even impose the assumption that the conformal structures induced from the immersions stay in a compact domain of the Teichmüller space, the sequence F_n would subsequentially converge only to a *branched* immersion on each component in the bubble tree convergence. True branch points might exist for the limiting map F_∞ . On the other hand, if $\Lambda < 16\pi$ in Theorem 1.2 then the Willmore energy of each immersion F_n is less than 8π for shrinkers (cf. [4]), hence by an inequality of Li-Yau (Theorem 6 in [18]), F_n must be an embedding and in turn Σ is a torus since it is Lagrangian.

When Σ has genus one, we can drop the assumption on the convergence of conformal structures in Theorem 1.2. Moreover, if the area bound is strictly less than 16π , the limiting surface has no branched points and we have smooth compactness.

Theorem 1.3. *Let $F_n : T \rightarrow \mathbb{C}^2$ be a sequence of Lagrangian self-shrinking tori with a uniform area upper bound Λ . Then (i) there is a subsequence of $\{F_n\}$ that converges smoothly to a branched Lagrangian self-shrinking torus $F_\infty : T \rightarrow \mathbb{C}^2$, and (ii) when $\Lambda < 16\pi$, F_∞ is an embedding.*

Our strategy of proving Theorem 1.3 is to rule out the possibility of degeneration of the conformal structures induced by F_n on T . To do this we need the general bubble tree convergence results in [5], [6] that allow the conformal structures degenerate. The key observation is that when the conformal structures degenerate, some homotopically nontrivial closed curves in the torus are pinched to points. Thus the limiting surface is a finite union of spheres (arising from collapse of the closed curves and from the bubbles at the energy concentration points) which are branched Lagrangian shrinkers, but Theorem 1.1 forbids this. The desired result then follows from Theorem 1.2.

2. PRELIMINARIES

Given an immersion $F : \Sigma \rightarrow \mathbb{R}^{m+k}$, let g be the induced metric $F^*\langle \cdot, \cdot \rangle$ on Σ . In local coordinates (x^1, \dots, x^m) of Σ , the second fundamental form A and the mean curvature vector \vec{H} are given by

$$A_{ij} = (\partial_{ij} F)^\perp, \quad \vec{H} = \sum_{i,j=1}^m g^{ij} A_{ij} = \Delta_g F$$

where Δ_g is the Laplace-Beltrami operator in the induced metric g .

Let G be a metric on \mathbb{R}^{m+k} defined by

$$(2.1) \quad G(\cdot, \cdot) = e^{-\frac{|x|^2}{m}} \langle \cdot, \cdot \rangle.$$

The following lemma is well-known and is proved by Angenent [1] in the case of hypersurface. The proof can be generalized to arbitrary codimension.

Lemma 2.1. *The immersion F satisfies equation (1.1) if and only if F is a minimal immersion with respect to the metric G (2.1) on \mathbb{R}^{m+k} .*

Next we consider immersions $F : \Sigma \rightarrow \mathbb{C}^m$. We identify \mathbb{R}^{2m} with \mathbb{C}^m by $z^i = u^i + \sqrt{-1}v^i, i = 1, \dots, m$. The map F is called a Lagrangian immersion if $F^*\omega = 0$, where

$$\omega = \sum_{i=1}^m du^i \wedge dv^i$$

is the standard symplectic form on \mathbb{C}^m . Let J be the standard complex structure on \mathbb{C}^m . It is known that ω, J and $\langle \cdot, \cdot \rangle$ on \mathbb{C}^m are related by

$$(2.2) \quad \langle JX, JY \rangle = \langle X, Y \rangle, \quad \omega(X, Y) = \langle JX, Y \rangle$$

for any $X, Y \in T_x \mathbb{C}^m, x \in \mathbb{C}^m$. Thus F is Lagrangian if and only if J sends the tangent vectors of Σ to the normal vectors in \mathbb{C}^m .

In particular, by equation (2.2), $J\vec{H}(x)$ is tangent to $F(\Sigma)$ at x for any Lagrangian immersion F . The mean curvature form α_H is the 1-form on Σ defined by: for all $x \in \Sigma$ and $Y \in T_x \Sigma$,

$$(2.3) \quad \alpha_H(Y) = \omega(\vec{H}(x), (F_*)_x Y) = g(J\vec{H}(x), (F_*)_x Y).$$

Let d denotes the exterior differentiation of Σ . It is shown in [10] that α_H is closed:

$$(2.4) \quad d\alpha_H = 0.$$

If F is also a self-shrinker, we have the following equation for α_H ([4], p.1521). For the convenience of the reader, we include a proof.

Lemma 2.2. *Let F be a Lagrangian self-shrinker. Then α_H satisfies*

$$(2.5) \quad d^* \alpha_H = -\frac{1}{2} \alpha_H (\nabla_g |F|^2),$$

where d^* is the formal adjoint of d on Σ with respect to g .

Proof. In local coordinates,

$$\alpha_H = \sum_{i=1}^m (\alpha_H)_i dx^i$$

where the coefficients are given by

$$\begin{aligned} (\alpha_H)_i &= \langle J\vec{H}, \partial_i F \rangle \\ &= -\langle JF^\perp, \partial_i F \rangle \\ &= -\langle JF, \partial_i F \rangle. \end{aligned}$$

Note that we have used the self-shrinker equation (1.1) in the second equality. Now fix a point $p \in \Sigma$ and take the normal coordinates at p . At p , $g_{ij} = g^{ij} = \delta_{ij}$ and $\Gamma_{ij}^k = 0$. We now calculate at p ,

$$\begin{aligned} d^* \alpha_H &= - \sum_{i,j=1}^m g^{ij} (\alpha_H)_{i;j} \\ &= - \sum_{i=1}^m (\alpha_H)_{i,i} \\ &= \sum_{i=1}^m \partial_i \langle JF, \partial_i F \rangle \\ &= \sum_{i=1}^m (\langle J\partial_i F, \partial_i F \rangle + \langle JF, \partial_{ii}^2 F \rangle) \end{aligned}$$

Using $J\partial_i F \perp \partial_i F$ and $\partial_{ii}^2 F = A_{ii}$ for each i at p ,

$$\begin{aligned} d^* \alpha_H &= \sum_{i=1}^m \langle JF, A_{ii} \rangle \\ &= \langle JF, \vec{H} \rangle \\ &= -\langle F, J\vec{H} \rangle \\ &= -\alpha_H(F^\top) \\ &= -\frac{1}{2} \alpha_H(\nabla_g |F|^2), \end{aligned}$$

where F^\top is the tangential component of F and we have used

$$F^\top = \sum_{i=1}^m \langle F, \partial_i F \rangle \partial_i F = \frac{1}{2} \nabla_g |F|^2$$

at p . □

3. LAGRANGIAN CONFORMAL BRANCHED IMMERSION

From now on we consider $m = 2$. Let (Σ, g_0) be a smooth Riemann surface. A smooth map $F : \Sigma \rightarrow \mathbb{C}^2$ is called a branched conformal immersion if

(1) there is a discrete set $B \subset \Sigma$ such that

$$F : \Sigma \setminus B \rightarrow \mathbb{C}^2$$

is an immersion,

- (2) there is a function $\lambda : \Sigma \rightarrow [0, \infty)$ such that $g := F^*\langle \cdot, \cdot \rangle = \lambda g_0$ on Σ , where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on \mathbb{C}^2 and λ is zero precisely at B , and
- (3) the second fundamental form A on $\Sigma \setminus B$ satisfies $|A|_g \in L^2(K \setminus B, d\mu)$ for any compact domain K in Σ , where $|\cdot|_g$ and $d\mu$ are respectively the norm and the area element with respect to g (Note g defines a Riemannian metric on $\Sigma \setminus B$ but not on Σ).

Elements in B are called the branch points of F . A conformal branched immersion $F : \Sigma \rightarrow \mathbb{C}^2$ is called Lagrangian (resp. self-shrinking) if it is Lagrangian (resp. self-shrinking) when restricted to $\Sigma \setminus B$. Note that when F is Lagrangian, the mean curvature form α_H is defined only on $\Sigma \setminus B$.

The following proposition is the key result on removable singularity of α_H . Note that in this proposition we do not assume Σ to be closed.

Proposition 3.1. *Let $F : \Sigma \rightarrow \mathbb{C}^2$ be a branched conformal Lagrangian self-shrinker with the set of branch points B . Then there is a smooth one form $\tilde{\alpha}$ on Σ which extends α_H and $d\tilde{\alpha} = 0$ on Σ .*

Proof. The result is local, so it suffices to consider Σ to be the unit disc \mathbb{D} with a branch point at the origin only. Let (x, y) be the local coordinate of \mathbb{D} . We write $\alpha_H = adx + bdy$ for some smooth function a and b on the punctured disk $\mathbb{D}^* = \{z \in \mathbb{D} : z \neq 0\}$. Let

$$(3.1) \quad \operatorname{div}(\alpha_H) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}, \quad D = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$

be the divergence and the gradient with respect to δ_{ij} . As F is conformal, $g_{ij} = \lambda \delta_{ij}$, where $\lambda = \frac{1}{2}|DF|^2$ and δ_{ij} is the euclidean metric on \mathbb{D} . By restricting to a smaller disk if necessary, we assume that the image $|F|$ and λ are bounded. By equation (1.1) we have

$$|\vec{H}| \leq |F|,$$

where $|\cdot|$ is taken with respect to $\langle \cdot, \cdot \rangle$ on \mathbb{C}^2 . As $g = F^*\langle \cdot, \cdot \rangle$, using equation (2.3) we see that

$$|\alpha_H|_g \leq C \text{ on } \mathbb{D}^*.$$

Using $g_{ij} = \lambda \delta_{ij}$ and $n = 2$, we have

$$\nabla_g = \lambda^{-1} D \text{ and } d_g^* = -\lambda^{-1} \operatorname{div}.$$

Hence equation (2.5) is equivalent to

$$(3.2) \quad \operatorname{div}(\alpha_H) = \frac{1}{2} \alpha_H(D|F|^2).$$

Moreover, as $|\alpha_H|_g = \sqrt{\lambda^{-1}(a^2 + b^2)}$, we also have

$$|a|, |b| \leq |\alpha_H|_g \sqrt{\lambda} \leq C \sqrt{\lambda}.$$

Thus $|a|, |b|$ are bounded on \mathbb{D}^* . To simplify notations, let

$$P = \frac{1}{2} \alpha_H(D|F|^2).$$

Note that P is also bounded on \mathbb{D}^* .

Both equations (2.4) and (3.2) are satisfied pointwisely in \mathbb{D}^* . We now show that they are satisfied in the sense of distribution on \mathbb{D} . That is, for all test functions $\phi \in C_0^\infty(\mathbb{D})$,

$$(3.3) \quad \int_{\mathbb{D}} \alpha_H \wedge d\phi = 0$$

and

$$(3.4) \quad \int_{\mathbb{D}} \alpha_H(D\phi) dx dy = - \int_{\mathbb{D}} P\phi dx dy.$$

Note that all the integrands in equation (3.3) and (3.4) are integrable, since a, b and P are in $L^\infty(\mathbb{D})$.

First we show (3.3). Let $\psi_r \in C_0^\infty(\mathbb{D})$, $r < 1/2$, be a cutoff function such that $0 \leq \psi \leq 1$, $|D\psi| \leq 2/r$ and

$$\psi(x) = \begin{cases} 1 & \text{when } |x| \geq 2r, \\ 0 & \text{when } |x| \leq r. \end{cases}$$

Then $\phi\psi_r \in C_0^\infty(\mathbb{D}^*) \cap C^\infty(\mathbb{D})$. Using Stokes' theorem and equation (2.4), we have

$$0 = \int_{\mathbb{D}^*} d(\phi\psi_r \alpha_H) = \int_{\mathbb{D}^*} d(\phi\psi_r) \wedge \alpha_H.$$

This implies

$$(3.5) \quad \int_{\mathbb{D}^*} \psi_r \alpha_H \wedge d\phi = - \int_{\mathbb{D}^*} \phi \alpha_H \wedge d\psi_r.$$

Since $\psi_r \rightarrow 1$ on \mathbb{D}^* as $r \rightarrow 0$ and $\alpha_H, d\phi$ are bounded,

$$\lim_{r \rightarrow 0} \int_{\mathbb{D}^*} \psi_r \alpha_H \wedge d\phi = \int_{\mathbb{D}^*} \alpha_H \wedge d\phi = \int_{\mathbb{D}} \alpha_H \wedge d\phi$$

by Lebesgue's dominate convergence theorem. To estimate the right hand side of equation (3.5), note that as $d\psi_r$ has support on $\mathbb{D}_{2r} \setminus \mathbb{D}_r$, where \mathbb{D}_s denotes the disk of radius s . Hence

$$(3.6) \quad \left| \int_{\mathbb{D}} \phi \alpha_H \wedge d\psi_r \right| \leq \frac{4C \sup |\phi|}{r} \int_{\mathbb{D}_{2r} \setminus \mathbb{D}_r} dx = 12\pi C \sup |\phi| r \rightarrow 0$$

as $r \rightarrow 0$. Thus equation (3.3) holds.

To show equation (3.4), we use the same cutoff function ψ_r . Then $\phi\psi_r \in C_0^\infty(\mathbb{D}^*) \cap C^\infty(\mathbb{D})$. By the divergence theorem,

$$0 = \int_{\mathbb{D}^*} \operatorname{div}(\phi\psi_r \alpha_H) dx dy = \int_{\mathbb{D}^*} \alpha_H(D(\phi\psi_r)) dx dy + \int_{\mathbb{D}^*} \phi\psi_r \operatorname{div}(\alpha_H) dx dy.$$

Now we use equation (3.2) to conclude

$$(3.7) \quad - \int_{\mathbb{D}^*} P\phi\psi_r dx dy = \int_{\mathbb{D}^*} \psi_r \alpha_H(D\phi) dx dy + \int_{\mathbb{D}^*} \phi \alpha_H(D\psi_r) dx dy.$$

Similarly, we can estimate the second term on the right hand side of equation (3.7) as for equation (3.6):

$$(3.8) \quad \left| \int_{\mathbb{D}^*} \phi \alpha_H(D\psi_r) dx dy \right| \leq 12\pi C \sup |\phi| r.$$

Using Lebesgue's dominate convergence theorem again, we can set $r \rightarrow 0$ in equation (3.7) to arrive at equation (3.4).

Writing $\alpha_H = adx + bdy$, equations (3.3) and (3.4) are equivalent to

$$(3.9) \quad \int_{\mathbb{D}} \left(a \frac{\partial \phi}{\partial y} - b \frac{\partial \phi}{\partial x} \right) dx dy = 0,$$

$$(3.10) \quad \int_{\mathbb{D}} \left(a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial y} \right) dx dy = - \int_{\mathbb{D}} P \phi dx dy,$$

for any test functions $\phi \in C_0^\infty(\mathbb{D})$.

For any $\psi \in C_0^\infty(\mathbb{D})$, set $\phi = \frac{\partial \psi}{\partial y}$ in equation (3.9), $\phi = \frac{\partial \psi}{\partial x}$ in equation (3.10) and cancel the cross term $b \frac{\partial^2 \psi}{\partial x \partial y}$ by taking summation of the two, we have

$$\int_{\mathbb{D}} a \Delta \psi dx dy = - \int_{\mathbb{D}} P \frac{\partial \psi}{\partial x} dx dy$$

where Δ is the Laplace operator in the euclidean metric on \mathbb{D} .

Similarly, set $\phi = \frac{\partial \psi}{\partial x}$ in equation (3.9), $\phi = \frac{\partial \psi}{\partial y}$ in equation (3.10) and take the difference of the two equations, we obtain

$$\int_{\mathbb{D}} b \Delta \psi dx dy = - \int_{\mathbb{D}} P \frac{\partial \psi}{\partial y} dx dy.$$

We conclude now that a and b satisfy

$$(3.11) \quad \begin{aligned} \Delta a &= \frac{\partial P}{\partial x}, \\ \Delta b &= \frac{\partial P}{\partial y} \end{aligned}$$

on \mathbb{D} in the sense of distribution. Now we apply the elliptic regularity theory for distributional solutions. As F is smooth and $\alpha \in L^\infty(\mathbb{D})$, we have $P \in L^2(\mathbb{D})$. Hence the right hand side of equation (3.11) is in $H_{-1}^{loc}(\mathbb{D})$. By the local regularity theorem ([11], Theorem 6.30), we have $a, b \in H_1^{loc}(\mathbb{D})$. This implies $P \in H_1^{loc}(\mathbb{D})$. Using this, we see that the right hand side of equation (3.11) is in $H_0^{loc}(\mathbb{D})$. By the same local regularity theorem again, these implies $a, b \in H_2^{loc}(\mathbb{D})$. Thus we can iterate this argument and see that $a, b \in H_s^{loc}(\mathbb{D})$ for all integers s . By the Sobolev embedding theorem we have $a, b \in C^\infty(\mathbb{D})$. Hence α_H can be extended to a smooth one form $\tilde{\alpha}$ on \mathbb{D} and $d\tilde{\alpha} = 0$ is satisfied on \mathbb{D} . \square

Now we proceed to the proof of Theorem 1.1.

Proof. Let $F : \mathbb{S}^2 \rightarrow \mathbb{C}^2$ be a branched conformal Lagrangian self-shrinker with branch points b_1, \dots, b_k . By Proposition 3.1, there is a smooth 1-form $\tilde{\alpha}$ on \mathbb{S}^2 such that $\tilde{\alpha} = \alpha_H$ on $\mathbb{S}^2 \setminus B$. As $\tilde{\alpha}$ is closed and the first cohomology group of \mathbb{S}^2 is trivial, there is a smooth function f on \mathbb{S}^2 such that $df = \tilde{\alpha}$. By equation (2.2), f satisfies

$$(3.12) \quad \Delta_g f = \frac{1}{2} df(\nabla_g |F|^2)$$

on $\mathbb{S}^2 \setminus B$. Note that this equation is elliptic but not uniformly elliptic on $\mathbb{S}^2 \setminus B$. By the strong maximum principle, the maximum of f cannot be attained in $\mathbb{S}^2 \setminus B$ unless f is constant. Let $b \in B$ be a point where f attains its maximum. Let \mathbb{D} be a local chart around b such that $g_{ij} = \lambda \delta_{ij}$ on \mathbb{D}^* . As $\Delta_g = \lambda^{-1} \Delta$ and $\nabla_g = \lambda^{-1} D$, equation (3.12) can be written

$$(3.13) \quad \Delta f = \frac{1}{2} df(D|F|^2) \quad \text{on } \mathbb{D}^*.$$

As f is smooth on \mathbb{D} , equation (3.13) is in fact satisfied on \mathbb{D} . By the strong maximum principle, f is constant as f has an interior maximum at b . Hence $\alpha_H = 0$ and $\vec{H} = 0$. This implies that F is a branched minimal immersion in \mathbb{C}^2 , which is not possible as \mathbb{S}^2 is compact. \square

4. PROOF OF THEOREM 1.2

Let F_n be a sequence of Lagrangian self-shrinkers which satisfies the hypothesis in the theorem. We need the following two lemmas for proving convergence of F_n .

The first lemma asserts that the images of F_n , for all n , stay in a bounded region whose size depends only on the area bound Λ . In the following we use B_r to denote the open ball in \mathbb{C}^2 with radius r centered at the origin.

Lemma 4.1. *Let F_n be a sequence of two dimensional self-shrinkers in \mathbb{R}^4 with a uniform area upper bound. Then the image of F_n lies in B_{R_0} , where R_0 depends only on the area upper bound.*

Proof. It is proved in [4] that for any m dimensional self-shrinker $F : \Sigma^m \rightarrow \mathbb{R}^{m+k}$.

$$(4.1) \quad \Delta_g |F|^2 = -2|F^\perp|^2 + 2m.$$

When Σ is compact, there is a point $p \in \Sigma$ where $|F|^2$ attains its minimum. Then we have

$$\nabla_g |F|^2 = 2F^\top = 0 \quad \text{and} \quad \Delta_g |F|^2 \geq 0 \quad \text{at } p.$$

This implies

$$0 \leq -2|F^\perp|^2 + 2m \quad \text{at } p.$$

Hence $|F|^2 \leq m$ at p . Thus each F_n must intersect the ball with radius $\sqrt{2}$.

On the other hand, we integrate equation (4.1) and use equation (1.1) to get (also derived in [4])

$$(4.2) \quad \mathcal{W}(F_n) := \frac{1}{4} \int_\Sigma |H_n|^2 d\mu_n = \frac{1}{2} \mu(F_n),$$

where $\mu(F_n)$ is the area of the immersion $F_n : \Sigma \rightarrow \mathbb{R}^{m+k}$.

Next we use Simon's diameter estimate ([20], see also (2.1) in [17]), which says that there is a constant C such that

$$(4.3) \quad \left(\frac{\mu(F_n)}{\mathcal{W}(F_n)} \right)^{\frac{1}{2}} \leq \text{diam} F_n(\Sigma) \leq C(\mu(F_n)\mathcal{W}(F_n))^{\frac{1}{2}}.$$

Here $\text{diam} F_n(\Sigma)$ is the extrinsic diameter given by

$$\text{diam} F_n(\Sigma) := \sup_{x, y \in \Sigma} |F_n(x) - F_n(y)|.$$

As the areas are uniformly bounded, the images of F_n all lie in B_{R_0} for some R_0 depending only on the area upper bound. \square

Let $U = B_{R_0+1}$ endowed with the metric G given by

$$(4.4) \quad G = e^{-\frac{|x|^2}{2}} \langle \cdot, \cdot \rangle.$$

The next lemma enables us to apply the results in [19] for harmonic maps into a compact Riemannian manifold.

Lemma 4.2. *There is a compact Riemannian manifold (N, \bar{g}) such that (U, G) is isometrically embedded into (N, \bar{g}) .*

Proof. Let $d = \frac{1}{R_0+1}$ and N is the disjoint union of B_{R_0+2} and B_d , with the identification that $x \sim y$ if and only if $y = \frac{x}{|x|^2}$ by the inversion. The manifold N is compact, as it can be identified as the one point compactification of \mathbb{C}^2 via the stereographic projection. Let g_1 be any metric on B_d . Let $\rho_1, \rho_2 \in C^\infty(N)$ be a partition of unity subordinate to the open cover $\{B_{R_0+2}, B_d\}$ in N and define a Riemannian metric on N by

$$\bar{g} = \rho_1 G + \rho_2 g_1.$$

As

$$B_{R_0+2} \cap B_d = \{x \in B_{R_0+2} : R_0 + 1 < |x| < R_0 + 2\},$$

$\bar{g} = \rho_1 G + \rho_2 g_1 = G$ on $B_{R_0+1} \subset B_{R_0+2}$. Thus the inclusion $U \subset B_{R_0+1} \subset N$ is an isometric embedding of U . \square

Now we are ready to prove Theorem 1.2.

Proof. As $F_n(\Sigma) \subset U$ for all n , we also treat F_n as a map with image in N . As F_n is a conformal minimal immersion with respect to the metric h_n on Σ and $G = \bar{g}$ on U , $F_n : (\Sigma, h_n) \rightarrow (N, \bar{g})$ is a sequence of harmonic maps. The area of $F_n(\Sigma)$ in (N, \bar{g}) is

$$\tilde{\mu}(F_n) = \int_{\Sigma} e^{-\frac{|F_n|^2}{2}} d\mu_{F_n^* \langle \cdot, \cdot \rangle}.$$

Therefore, as $e^{-\frac{|F_n|^2}{2}} \leq 1$ we have

$$\tilde{\mu}(F_n) \leq \mu(F_n) < \Lambda.$$

As $F_n : (\Sigma, h_n) \rightarrow (N, \bar{g})$ is conformal, the area $\tilde{\mu}(F_n)$ in N is the same as the Dirichlet energy:

$$E_{h_n, \bar{g}}(F_n) = \tilde{\mu}(F_n) < \Lambda.$$

By assumption, the sequence of metrics h_n converges to h . Hence we can apply the theory of bubble tree convergence of harmonic maps developed in [19]. In particular, by passing to subsequence if necessary, there is a finite (possibly empty) set $S \subset \Sigma$ and a harmonic map $F_\infty : (\Sigma, h) \rightarrow (N, \bar{g})$ such that $F_n \rightarrow F_\infty$ locally in $C^\infty(\Sigma \setminus S)$ ¹. For each $x \in S$, there is a finite sequence of nontrivial harmonic maps $s_{x,i} : \mathbb{S}^2 \rightarrow N$, $i = 1, \dots, n_x$ such that the renormalized maps $F_{n,I}$ (See [19] for the details) of F_n converges to the harmonic bubble tree map formed by F_∞ and $s_{x,i}$. In particular, we have the Hausdorff convergence

$$F_n(\Sigma) \rightarrow F_\infty(\Sigma) \cup \bigcup_{x \in S} \bigcup_i s_{x,i}(\mathbb{S}^2).$$

As a result, F_∞ and $s_{x,i}$ all have image in $\bar{U} \subset N$.

The harmonic maps $s_{x,i} : \mathbb{S}^2 \rightarrow \bar{U}$ are branched conformal minimal immersions. From the construction of bubble tree convergence [19], a bubble at $x \in \Sigma$ is constructed by considering the renormalization

$$\tilde{f}_n(z) := f_n(\exp_x(\lambda_n z + c_n)),$$

where $c_n, \lambda_n \rightarrow 0$ as $n \rightarrow \infty$. As the normalization is only on the domain, \tilde{f}_n are also Lagrangian. The regularity results in [23] show that there is a finite set $\tilde{S} \subset \mathbb{C}$ such that \tilde{f}_n converges locally in $C^\infty(\mathbb{C} \setminus \tilde{S})$ to a harmonic map $\tilde{f}_\infty : \mathbb{S}^2 \rightarrow N$. Thus the bubble \tilde{f}_∞ is Lagrangian. This applies to each level of the bubble tree, so all bubbles formed are also Lagrangian. Using Lemma 4.2 and Lemma 2.1, each $s_{x,i}$ is a branched Lagrangian self-shrinker in \mathbb{C}^2 . By Theorem 1.1, such an immersion does not exist. Thus there is no bubble formed in the convergence process (that is, S is empty), and the convergence $F_n \rightarrow F_\infty$ is in $C^\infty(\Sigma)$.

By Theorem 3.3 in [23], as each $F_n : \Sigma \rightarrow N$ is nontrivial by definition, there is an $\epsilon > 0$ such that $E_{h_n, \bar{g}}(F_n) \geq \epsilon$ for all $n \in \mathbb{N}$. Using the energy identity [16], we have

$$(4.5) \quad E_{h_n, \bar{g}}(F_n) \rightarrow E_{h, \bar{g}}(F_\infty) \quad \text{as } n \rightarrow \infty.$$

Hence $E_{h, \bar{g}}(F_\infty) \geq \epsilon$ and F_∞ is nontrivial (One can also use the estimate of diameter and the Hausdorff convergence as in the proof of Theorem 1.3 in the next section to show that F_∞ is nontrivial).

As the convergence $F_n \rightarrow F_\infty$ is in $C^\infty(\Sigma)$ and the metrics h_n converges to h smoothly, the harmonic map $F_\infty : (\Sigma, h) \rightarrow (N, \bar{g})$ is also conformal. Thus, F_∞ is a branched conformal minimal immersion from (Σ, h) into (N, \bar{g}) . It is also a Lagrangian immersion. Since the image of F_∞ lies in \bar{U} , we also view F_∞ as a branched Lagrangian minimal immersion in \mathbb{C}^2 with respect to the metric G defined by (4.4). By Lemma 2.1, F_∞ is then a Lagrangian branched conformal self-shrinker. \square

¹In [24], it is shown that the convergence is in locally $C^1(\Sigma \setminus S)$. As each $F_n : (\Sigma, h_n) \rightarrow (\mathbb{C}^2, G)$ is harmonic and $h_n \rightarrow h$, by the standard elliptic estimates (Chapter 6 of [14]) and a bootstrapping argument, there are constants $C(m)$ such that

$$\|F_n\|_{C^m} \leq C(m)$$

for all $n \in \mathbb{N}$. Using Arzela-Ascoli Theorem and picking a diagonal subsequence, one shows that a subsequence of $\{F_n\}$ converges locally smoothly to a smooth mapping $\Sigma \rightarrow \mathbb{C}^2$, which is F_∞ .

5. PROOF OF THEOREM 1.3

Proof. Let $\{F_n\}$ be a sequence of immersed Lagrangian self-shrinking tori in \mathbb{C}^2 with a uniform area upper bound. Let h_n be the metric on the torus T which is conformal to $F_n^*\langle \cdot, \cdot \rangle$ and with zero Gauss curvature. If we can show that h_n stays in a bounded domain of the Teichmüller space, then (i) follows from Theorem 1.2.

Using Lemma 4.1, Lemma 4.2 and Lemma 2.1, we also treat each F_n as a minimal immersion in N , this means that $F_n : (T, h_n) \rightarrow (N, \bar{g})$ is conformal and harmonic. Assume the contrary that the conformal structures degenerate. In this case, there is a mapping \hat{F}_∞ from Σ_∞ to N and the image $F_n(T)$ converges in the Hausdorff distance to $\hat{F}_\infty(\Sigma_\infty)$ in N . Here Σ_∞ is a stratified surfaces $\Sigma_\infty = \Sigma_0 \cup \Sigma_b$ formed by the *principal component* Σ_0 and *bubble component* Σ_b . The principal component Σ_0 is formed by pinching several closed, homotopically nontrivial curves in T and the bubbling component is a union of spheres. There are no necks between the components since F_n is conformal. The map \hat{F}_∞ is continuous on Σ_∞ and harmonic when restricted to each component of Σ_∞ . Since all the components intersect each others possibly at finitely many points, \hat{F}_∞ is harmonic except at a finite set \hat{S} .

Since the conformal structures determined by the metrics h_n degenerate, at least one homotopically nontrivial closed curve must be pinched to a point as $n \rightarrow \infty$. It follows that Σ_0 is a finite union of \mathbb{S}^2 's. Each of these 2-spheres is obtained by adding finitely many points to the cylinder $\mathbb{S}^1 \times \mathbb{R}$ that comes from pinching one or two homotopically nontrivial loops: two at the infinity and at most finitely many at the blowup points of the sequence F_n , by the removable singularity theorem of Sacks-Uhlenbeck [23]. Therefore, \hat{F}_∞ is a finite union of harmonic mappings \hat{F}_∞^i from the sphere to N .

Since F_n are conformal, there are no necks between the components. The bubble tree convergence described above are given by the results in [6] or [5]. In [6], the limiting surface is a stratified surface with geodesics connecting the two dimensional components. But together with conformality of each F_n and Proposition 2.6 in [6], one sees that all the geodesics involved have zero length. Alternatively, we can use the compactness theorem in [5] which says: Suppose that $\{f_k\}$ is a sequence of $W^{2,2}$ branched conformal immersions of (Σ, h_k) in a compact manifold M . If

$$\sup_k \{\mu(f_k) + W(f_k)\} < +\infty$$

then either $\{f_k\}$ converges to a point, or there is a stratified sphere Σ_∞ and a $W^{2,2}$ branched conformal immersion $f_\infty : \Sigma_\infty \rightarrow M$, such that a subsequence of $\{f_k(\Sigma)\}$ converges to $f_\infty(\Sigma_\infty)$ in the Hausdorff topology, and the area and the Willmore energy satisfy

$$\mu(f_\infty) = \lim_{k \rightarrow +\infty} \mu(f_k) \text{ and } W(f_0) \leq \lim_{k \rightarrow +\infty} W(f_k).$$

The conditions of the theorem are satisfied by the sequence $\{F_n\}$ as the area $\tilde{\mu}(F_n)$ and the Willmore energy $W(F_n)$ in N (which is zero as each F_n is minimal immersion in N) are uniformly bounded. In our situation, F_k will not converges to a point since $\text{diam}(F_n(T)) \geq \sqrt{2}$ by equations (4.3) and (4.2). Thus the inequality on the Willmore

energies in N

$$W(\hat{F}_\infty) \leq \lim_{n \rightarrow \infty} W(F_n) = 0$$

implies that the limiting \mathbb{S}^2 's are all branched minimal surfaces in N .

Consequently, the images $F_n(T)$ converge in the Hausdorff distance to the image of finitely many harmonic maps $\mathbb{S}^2 \rightarrow N$. These harmonic maps are conformal branched immersions, which are also Lagrangian by similar reasons as in the proof of Theorem 1.2. By Theorem 1.1, all these harmonic maps are trivial. Hence, the images $F_n(T)$ converge in the Hausdorff distance to a point in \mathbb{C}^2 . Again, this is impossible by the diameter estimate $\text{diam}(F_n(T)) \geq \sqrt{2}$. This contradiction shows that the conformal structures cannot degenerate and that finishes the proof of (i).

To show (ii), we assume $F_n \rightarrow F_\infty$ smoothly by (i). Note that when the area upper bound $\Lambda < 16\pi$, by equation (4.2), the Willmore energy $\mathcal{W}(F_n)$ of F_n in $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ admits an upper bound $\Lambda/2 < 8\pi$. By Theorem 1 in [5],

$$\mathcal{W}(F_\infty) \leq \lim_{n \rightarrow \infty} \mathcal{W}(F_n),$$

thus the limiting Lagrangian shrinker F_∞ also has Willmore energy strictly less than 8π . By Proposition 4.1 in [17] (also, Theorem 3.1 there), the limiting mapping $F_\infty : \Sigma \rightarrow \mathbb{C}^2$ has no branch points, therefore it is an immersion. By Theorem 6 in [18], the immersion F_∞ must be an embedding. \square

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